Indian Statistical Institute
First Semester Examination 2004-2005
B.Math (Hons.) III Year
Complex Analysis

Time: 4 hrs   Date:22-11-04   Max. Marks : 100

Note : The question paper carries a total of 125 marks. Answer as much as you can. But the maximum you may actually score is 100.

1. (a) Let $\sum\limits_{n=0}^{\alpha} a_n z^n$ be the Taylor expansion around 0 of an analytic function f on the disc with centre 0 and radius $R > 0$. Show that for $0 < r < R$, $\frac{1}{2\pi}\int\limits_0^{s\pi} |f(re^{i\theta})|^2 d\theta = \sum\limits_{n=0}^{\alpha} |a_n|^2 r^{2n}$.

   (b) Use part (a) to prove the maximum modulus principle.

   (c) Let f be an entire function which maps $\{z : |z| > 1\}$ into itself. Use the previous parts to prove that f is a polynomial. [5+5+15 =25]

2. Let f be an analytic function on the planar domain Ω.

   (a) Fix $z_0 \in \Omega$. Suppose $f'(z_0) \neq 0$. Then show that f is one-one on some neighbourhood of z_0.

   (b) If f is one-one on Ω then show that $f'(z) \neq 0$ for all $z \in \Omega$.

   (c) If $f'(z) \neq 0$ for all $z \in \Omega$, does it follow that f is one-one? Justify your answer. [10+10+5 =25]

3. Let Ω be a planar domain which intersects the real axis $I\!R$. Let f be an analytic function on Ω such that $f(\Omega \cap I\!R) \subseteq I\!R$.

   (a) Show that f has an analytic continuation to $\Omega \cup \bar{\Omega}$.

   (b) Prove that the zeroes of the analytic continuation occur in conjugate pairs. [20+5=25]

4. (a) Define Euler's Gamma function on the half-plane $\{Re(z) > 0\}$ by the usual integral formula. Prove that it satisfies $\Gamma(z+1) = z\Gamma(z)$.

   (b) Use (a) to prove that Γ has a meromorphic continuation to the entire complex plane. [5+5+15 =25]

   (c) Show that Γ has simple poles at the non-positive integers and compute its residues at all these poles.

5. Fix $0 < \epsilon < 1 < R$. Let $\gamma_{\epsilon,R}$ be the closed path which traverses the circle with centre 0 and radius ϵ once in the anticlockwise direction, then goes from ϵ to R along a straight line, then traverses the circle with centre 0 and radius R once in the clockwise direction, and finally goes from R to ϵ along a straight line. For any complex number s with

$0 < Re(s) < 1$, consider the integral $\int\limits_{\gamma_{\epsilon,R}} \frac{x^{s-1}}{x+1}dx$. Here the integrand is defined, using the branch of log with singularities on $I\!R^+$, to be continuous on the path.

(a) Use the residue theorem to compute this integral.

(b) Examine what happens to the four parts of the integral when $\epsilon \searrow 0,\ R \nearrow \infty$. Hence prove the formula

$$\int\limits_0^\infty \frac{x^{s-1}}{x+1}dx = \frac{\pi}{\sin \pi s} \quad (0 < Re(s) < 1)$$

[5+20 =25]